TENSOR PRODUCTS OF BANACH ALGEBRAS WITH INVOLUTION

BY

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1. **Introduction.** The present paper deals with tensor products of Banach algebras with involution. In a sense this paper may be considered an investigation of the extent to which certain properties of Banach-*-algebras are passed on to their tensor products.

In §§2 and 3 we present much of the background material needed throughout the paper. In §2 we gather some facts from the general representation theory and we introduce the notions of the spectrum and the enveloping algebra of a Banach-*-algebra. In §3 we turn our attention to tensor products, specifically the problem of norming a given tensor product.

§4 deals with the enveloping algebra of a tensor product. We show that with certain conditions on two Banach-*-algebras and on the norm norming their tensor product the enveloping algebra of their tensor product can be identified with a tensor product of their respective enveloping algebras. The main result of this section generalizes a theorem by Okayasu [16, Theorem 3].

In the following section we relate the spectrum of a tensor product to the product of the spectra of the Banach-*-algebras in question. Without restrictions on these algebras the latter product will be a subset of the former, but if one of the algebras is of type I, then the two sets can be identified. §5 is to a large extent a natural extension of work done by Wulfsohn [25], although connections also exist with results due to Okayasu and Takesaki [16], [17], to Gelbaum [4], [5], [6] and Gil de Lamadrid [7], [8].

On the basis of the theory developed up to this point we consider in §6 the concept of *-semisimplicity. We show that a tensor product of *-semisimple Banach-*-algebras is *-semisimple if and only if a variation of a condition of approximation introduced by Grothendieck [9] is satisfied.

The theory developed in this paper finds a natural application in the study of certain vector valued group algebras. The last section deals with this. We consider group algebras of functions from a compact T_2 group to Banach-*-algebras. The enveloping algebra and the spectra of such algebras are identified. Moreover, it is shown that *-semisimplicity of the range algebra is passed on to the group algebra

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in question. This section can be considered a continuation of work done by Spicer [20], [21].

2. Background. Representations, the spectrum, the enveloping algebra. In this section we collect much of the theory of algebras with involution which will be needed later. We remark at the outset that we shall always use single bars $|\cdot|$ to denote the norm in a normed linear space. Moreover, to avoid confusion with the commutant of a set we shall use the notation E_* to denote the dual space of a normed linear space E.

2.1. A representation μ (sometimes $*$-representation) of a $*$-algebra A is an involution-preserving homomorphism of A into the C^*-algebra $L(H_\mu)$ of all bounded linear operators on some Hilbert space. If the representation is an isomorphism, i.e. 1-1, we say it is *faithful*.

2.2. A representation μ on a Hilbert space H_μ is *topologically irreducible* if H_μ and $\{0\}$ are the only closed subspaces of H_μ left invariant by $\mu(A)$.

The von Neumann algebra (or w^*-algebra) generated by a set $B \subset L(H)$ will be denoted by $w^*(B)$ and the commutant of B will be denoted by B'. We also use the notation

$$C_H = \{\lambda I; \lambda \in C\} \subset L(H).$$

Here C is the complex field and I is the identity operator on H.

Using this notation we can list some characterizations of topological irreducibility; most of these are proved in [3, Proposition 2.3.1].

PROPOSITION 2.3. *If A is a $*$-algebra and μ a representation of A in a Hilbert space H_μ then the following conditions are equivalent*:

(i) *μ is topologically irreducible.*

(ii) *$[\mu(A)]' = C_{H_\mu}$.*

(iii) *Every nonzero vector h in H_μ is cyclic for μ, i.e. $H_\mu = \{\mu(x)h; x \in A\}^-$ where $^-$ denotes norm closure.*

(iv) *$[\mu(A)]'' = L(H_\mu)$.*

(v) *$w^*[\mu(A)] = L(H_\mu)$.*

2.4. The spectrum of a $*$-algebra A, denoted by $\hat{A}$ or $A^\wedge$, is the set of equivalence classes (relative to the unitary equivalence $\cong$ (see for example [3, p. 26])) of topologically irreducible representations on Hilbert space. If μ is topologically irreducible then $\bar{\mu}$ or $\mu^\wedge$ will denote the element of $\hat{A}$ to which μ belongs.

2.5. A continuous linear functional f on a normed $*$-algebra A is *positive* if $f(x^*x) \geq 0$ for all $x \in A$. f is a *state* if $|f| = 1$ and *pure* (or indecomposable) if for any positive continuous linear functional g such that $f - g$ is positive it follows that g is a multiple of f.

2.6. An algebra A with involution is of type I if all representations of A are of type I (see [3, Definition 5.4.2]).

2.7. If we denote by $P(A)$ the set of pure states on a Banach-*-algebra A with approximate identity then there is a natural mapping $\delta: P(A) \to \hat{A}$. δ is onto since if $\pi \in \bar{\pi} \in \hat{A}$ and p is a pure state associated with π (i.e. $p(x) = (\pi(x)\xi \mid \xi)$ for some $\xi \in H_\pi$) then the representation generated by p is equivalent to π [3, 2.4.6]. $P(A)$ is a subset of the unit ball in the dual space of A and thus has a natural topology, the relative weak-*-topology. We topologize $\hat{A}$ by giving it the quotient topology relative to δ, i.e. the strongest topology on $\hat{A}$ with respect to which δ is continuous. If A is a C^*-algebra then this definition yields the same topology as the one defined in terms of the Jacobson topology on the set of primitive ideals (see [3, p. 60 and §3.4]). In 2.9 we shall show that δ is open.

2.8. Let A be a Banach-*-algebra and let R be the set of representations of A. For each $x \in A$ define $|x|' = \sup_{\pi \in R} |\pi(x)|$.

Then $|\cdot|'$ is a seminorm on A with the properties (i) $|xy|' \le |x|'|y|'$, (ii) $|x^*|' = |x|'$, and (iii) $|x^*x|' = |x|'^2$ for all $x, y \in A$. The set

$$R^* = \{x \in A; |x|' = 0\}$$

is called the *-radical and will be studied in §6. Clearly, the completion of A/R^* with respect to the quotient norm of $|\cdot|'$ is a C^*-algebra. This C^*-algebra is the enveloping algebra of A and is denoted by $C^*(A)$. The natural mapping $\varepsilon: A \to C^*(A)$ is called the enveloping mapping.

2.9. Let A be a Banach-*-algebra with approximate identity. By [3, 2.7.4] there is a 1-1 mapping $S: \hat{A} \to [C^*(A)]^\wedge$. Moreover, if $\varepsilon_*: (C^*(A))_* \to A_*$ is the adjoint mapping of the enveloping mapping ε, then the restriction $Q = \varepsilon_*|_{P(C^*(A))}$ is a homeomorphism. Here we shall show that S is a homeomorphism and that $\delta: P(A) \to \hat{A}$ is an open mapping.

PROPOSITION 2.10. *Let A be a Banach-*-algebra with approximate identity. Then $\delta: P(A) \to \hat{A}$ is open, while $S: \hat{A} \to [C^*(A)]^\wedge$ is a homeomorphism.*

Proof. Let $\delta_\varepsilon: P(C^*(A)) \to [C^*(A)]^\wedge$ be the canonical mapping corresponding to δ. δ_ε is open [3, 3.4]. We have the following commutative diagram

$$
\begin{array}{ccc}
P(A) & \xrightarrow{\ \delta\ } & \hat{A} \\
{\scriptstyle Q^{-1}}\downarrow & & \downarrow{\scriptstyle S} \\
P(C^*(A)) & \xrightarrow[\delta_\varepsilon]{} & [C^*(A)]^\wedge
\end{array}
$$

We first indicate the proof of the fact that S is open and continuous, i.e. a homeomorphism. Let $U \subset \hat{A}$ be open. Since $\hat{A}$ has the quotient topology relative to δ, $\delta^{-1}(U)$ is open and hence $Q^{-1}(\delta^{-1}(U))$ is open. Since $S(U) = \delta_\varepsilon(Q^{-1}(\delta^{-1}(U)))$, $S(U)$ is open, hence S is open. Conversely, if $V \subset [C^*(A)]^\wedge$ is open, then $\delta^{-1}(S^{-1}(V)) = Q(\delta_\varepsilon^{-1}(V))$ is open. Since $\hat{A}$ has the quotient topology $S^{-1}(V)$ is open. This shows that S is continuous and hence that S is a homeomorphism. From this it follows easily that δ is open: let $U \subset P(A)$ be open. Since $\delta(U) = S^{-1}(\delta_\varepsilon(Q^{-1}(U)))$ the result follows from the above. This completes the proof.

3. Tensor products, norms.

3.1. Let E_1, E_2 be normed linear spaces and $E_1 \otimes E_2$ their algebraic tensor product (see for example [19], or [9]). As is known there is no one canonical way in which norms on E_1, E_2 define a norm on $E_1 \otimes E_2$, although several natural norms on $E_1 \otimes E_2$ can be defined. Among these we mention *the least cross norm* λ, defined as follows. Given

$$t = \sum_{i=1}^{n} x_i \otimes y_i \in E_1 \otimes E_2, \qquad \lambda(t) = \sup \left| \sum_{i=1}^{n} x'(x_i) y'(y_i) \right|$$

where the sup is taken over the set of all x', y' in the respective unit balls of the dual spaces of E_1, E_2.

Another natural norm on $E_1 \otimes E_2$ is *the greatest cross norm* γ, which is defined in the following manner. For $t \in E_1 \otimes E_2$, $\gamma(t) = \inf \sum |x_i| \, |y_i|$ with the inf taken over all representatives $\sum_{i=1}^{n} x_i \otimes y_i = t$. The term 'cross norm' refers to the fact that

$$\lambda(x_1 \otimes x_2) = \gamma(x_1 \otimes x_2) = |x_1| \, |x_2|$$

for all $x_1 \in E_1$, $x_2 \in E_2$.

We say a norm c on $E_1 \otimes E_2$ is *ordinary* if $\lambda \leq c \leq \gamma$. If c is a norm on $E_1 \otimes E_2$ then the normed linear space thus arising will be denoted by $E_1 \otimes^c E_2$. The completion of $E_1 \otimes^c E_2$ with respect to c is denoted by $E_1 \otimes_c E_2$.

3.2. If E_1, E_2 are algebras there is a natural definition of multiplication in $E_1 \otimes E_2$ turning this into an algebra.

If $t_1 = \sum x_i \otimes y_i$ and $t_2 = \sum u_j \otimes v_j$ are elements in $E_1 \otimes E_2$, then

$$t_1 t_2 = \sum_{ij} x_i u_j \otimes y_i v_j.$$

That this is a meaningful concept has been proved by Turumaru [24]. A shorter proof is due to Gil de Lamadrid. We shall reproduce this proof here. It relies on the following

PROPOSITION [1, p. 4]. *Let* $M: E_1 \times E_2 \to E_1 \otimes E_2$ *be the natural mapping* $M(x, y) = x \otimes y$. *If G is any vector space and $f: E_1 \times E_2 \to G$ is a bilinear mapping, then there is a unique linear mapping $g: E_1 \otimes E_2 \to G$ so that $f = g \circ M$.*

With this proposition in mind we can easily prove that the above definition of multiplication is meaningful. Fix $X = \{x_i\}_{i=1}^{n} \subset E_1$, $Y = \{y_i\}_{i=1}^{n} \subset E_2$ and define $f_{XY}: E_1 \times E_2 \to E_1 \otimes E_2$ by

$$f_{XY}(x, y) = \sum_i xx_i \otimes yy_i.$$

f_{XY} is bilinear. Hence there is a unique linear mapping $g_{XY}: E_1 \otimes E_2 \to E_1 \otimes E_2$

with the property that $f_{XY}(x, y) = g_{XY}(M(x, y))$ for all $(x, y) \in E_1 \times E_2$, i.e.

$$g_{XY}\left(\sum_j u_j \otimes v_j\right) = g_{XY}\left(\sum_j M(u_j, v_j)\right)$$

$$= \sum_j g_{XY}(M(u_j, v_j)) = \sum_j f_{XY}(u_j, v_j)$$

$$= \sum_j \sum_i u_j x_i \otimes v_j y_i \quad \text{for any } \sum_j u_j \otimes v_j \in E_1 \otimes E_2.$$

Repetition of this argument with the fixed factors to the left yields the desired conclusion.

3.3 Whenever we consider $E_1 \otimes^c E_2$ or $E_1 \otimes_c E_2$, where E_1, E_2 are normed algebras we shall (explicitly or otherwise) assume that $E_1 \otimes^c E_2$ is a normed algebra, i.e. that c satisfies

$$c(t_1 t_2) \leqq c(t_1)c(t_2) \quad \text{for all } t_1, t_2 \in E_1 \otimes E_2.$$

We say that such a c is *compatible with multiplication on* $E_1 \otimes E_2$. The greatest cross norm γ is always compatible with multiplication [4, Theorem 1] while there are examples of algebras on which λ is not compatible with multiplication [7, §5]. Under certain circumstances, however, λ does yield a Banach algebra [8], [22].

3.4. If E_1, E_2 are *-algebras, then an involution can be defined on $E_1 \otimes E_2$ in this fashion:

$$\text{if } t = \sum x_i \otimes y_i \in E_1 \otimes E_2, \text{then } t^* = \sum x_i^* \otimes y_i^*.$$

A method similar to that employed in 3.2 will show that * as defined here is a well-defined involution. A norm c on $E_1 \otimes E_2$ is *compatible with involution on* $E_1 \otimes E_2$ if $c(t) = c(t^*)$ for all $t \in E_1 \otimes E_2$. Both λ and γ are compatible with involution.

3.5. If E_1, E_2 are C^*-algebras then there are several norms c that turn $E_1 \otimes_c E_2$ into a C^*-algebra. We shall pay special attention to two such norms.

The least C^*-norm α. Let E_1, E_2 be C^*-algebras and

$$\mu_1: E_1 \to L(H_1), \qquad \mu_2: E_2 \to L(H_2)$$

be representations. If $t = \sum x_i \otimes y_i \in E_1 \otimes E_2$, then we define

$$\mu_1 \otimes \mu_2(t) = \sum \mu_1(x_i) \otimes \mu_2(y_i) \in L(H_1) \otimes L(H_2) \subset L(H_1 \otimes_\sigma H_2)$$

where σ is the canonical norm induced by the product of the inner products on H_1 and H_2. α is defined as follows:

$$\alpha(t) = |\mu_1 \otimes \mu_2(t)| \quad \text{for any } t \in E_1 \otimes E_2$$

where μ_1, μ_2 are faithful representations. Wulfsohn [25] shows that this definition is independent of the choice of representations.

The greatest C^*-norm ν. This norm was introduced by Guichardet [11] and defined by $\nu(t) = \sup_\pi |\pi(t)|$ where π ranges over the set of all representations of $E_1 \otimes E_2$ satisfying the condition

$$|\pi(x_1 \otimes x_2)| \leqq |x_1| |x_2| \quad \text{for all } x_i \in E_i (i = 1, 2).$$

Clearly $\alpha \leq \nu$. The question of equality of the two norms has been studied by Takesaki [23]. He says that a C^*-algebra A has property (T) if $A \otimes_\alpha B = A \otimes_\nu B$ for any C^*-algebra B and shows that the class of C^*-algebras with property (T) properly contains the class of C^*-algebras of type I.

We remark that both α and ν are cross norms. From the definition of ν and the fact that $\alpha \leq \nu$ it follows that it suffices to show this for α; and for α this is a consequence of [8, Theorem 1.2].

4. The enveloping algebra of a tensor product.

4.1. In this section we show that in a great many cases the enveloping algebra of a tensor product can be identified with a tensor product of the respective enveloping algebras of the Banach-*-algebras in question.

But first we state and prove a result concerning extension of certain representations. This result will be helpful later.

4.2. As we indicated in 3.5, if E_1, E_2 are Banach-*-algebras and $\mu_i: E_i \to L(H_{\mu_i})$ representations, then we can define

$$\mu_1 \otimes \mu_2: E_1 \otimes E_2 \to L(H_{\mu_1} \otimes_\sigma H_{\mu_2})$$

by

$$\mu_1 \otimes \mu_2\Big(\sum x_i \otimes y_i\Big) = \sum \mu_1(x_i) \otimes \mu_2(y_i) \in L(H_{\mu_1} \otimes_\sigma H_{\mu_2})$$

for any $\sum x_i \otimes y_i \in E_1 \otimes E_2$. If c is a norm on $E_1 \otimes E_2$ then $\mu_1 \otimes \mu_2$ can be extended to all of $E_1 \otimes_c E_2$ under rather mild restrictions on c. We have the following

PROPOSITION 4.3. *Let E_1, E_2 be Banach-*-algebras with approximate identities and c be an ordinary norm on $E_1 \otimes E_2$, compatible with multiplication and involution. If μ_i is a representation of E_i $(i=1, 2)$ then $\mu_1 \otimes \mu_2$ can be uniquely extended to $E_1 \otimes_c E_2$. The extended representation will also be denoted by $\mu_1 \otimes \mu_2$.*

The proof of this proposition requires the following

LEMMA 4.4. *Let E_1, E_2 and c be as in Proposition 4.3 and let p_i be a continuous positive linear functional on E_i $(i=1, 2)$. $p_1 \otimes p_2$ defined on $E_1 \otimes E_2$ by*

$$p_1 \otimes p_2\Big(\sum x_i \otimes y_i\Big) = \sum p_1(x_i)p_2(y_i)$$

for any $\sum x_i \otimes y_i \in E_1 \otimes E_2$ can be uniquely extended to a positive continuous linear functional on $E_1 \otimes_c E_2$ which will also be denoted by $p_1 \otimes p_2$.

The proof of this lemma is a straightforward adaptation to the present situation of the proof of [24, Lemma 1].

Proof of Proposition 4.3. Given any representation π of an algebra A there are representations π_0 and π_i such that $\pi = \pi_0 \oplus (\sum \oplus \pi_i)$ where π_0 is trivial and π_i is

cyclic [3, 2.2.6 and 2.2.7]. Using this decomposition we get that

$$\mu_1 \otimes \mu_2 = \left(\mu_0^1 \oplus \left(\sum \oplus \mu_i^1\right)\right) \otimes \left(\mu_0^2 \oplus \left(\sum \oplus \mu_j^2\right)\right)$$

$$= \sum_{ij} \oplus (\mu_i^1 \otimes \mu_j^2)$$

where each μ_i^1, μ_j^2 is either cyclic or trivial. Since it suffices to show that each $\mu_i^1 \otimes \mu_j^2$ can be extended we can assume without loss of generality that μ_1 and μ_2 are cyclic with cyclic vectors $a_1 \in H_{\mu_1}$ and $a_2 \in H_{\mu_2}$, respectively (obviously the trivial representation can be extended). Define

$$p_i(x_i) = (\mu_i(x_i)a_i \mid a_i) \quad \text{for all } x_i \in E_i \ (i = 1, 2).$$

By Lemma 4.4 $p_1 \otimes p_2$ defines a positive continuous linear functional on $E_1 \otimes_c E_2$. Let π' be the representation in a Hilbert space H generated by $p_1 \otimes p_2$ and let d be a cyclic vector for π' such that

$$p_1 \otimes p_2(t) = (\pi'(t)d \mid d) \quad \text{for all } t \in E_1 \otimes_c E_2$$

(see [3, Proposition 2.4.4]).

Since $E_1 \otimes^c E_2$ is dense in $E_1 \otimes_c E_2$ and d is cyclic

$$H_0 = \{\pi'(s)d; \ s \in E_1 \otimes^c E_2\}$$

is dense in H.

We shall define an isometry onto, $U: H \to H_{\mu_1} \otimes_\sigma H_{\mu_2}$. If $h \in H_0$, then $h = \pi'(s)d$ for some $s \in E_1 \otimes^c E_2$; we let

$$Uh = \mu_1 \otimes \mu_2(s)(a_1 \otimes a_2).$$

An easy computation shows that

$$|h|^2 = (\pi'(s)d \mid \pi'(s)d) = (\pi'(s*s)d \mid d)$$

$$= (\mu_1 \otimes \mu_2(s*s)(a_1 \otimes a_2) \mid (a_1 \otimes a_2))$$

$$= |Uh|^2$$

so that U is an isometry. To conclude that U can be extended to a surjective mapping it suffices to observe that

$$H_1 = \{\mu_1 \otimes \mu_2(s)(a_1 \otimes a_2); \ s \in E_1 \otimes^c E_2\}$$

is dense in $H_{\mu_1} \otimes_\sigma H_{\mu_2}$, since a_i is cyclic for μ_i in H_{μ_i} $(i = 1, 2)$.

We denote the extended isometry also by U and define

$$\pi: E_1 \otimes_c E_2 \to L(H_{\mu_1} \otimes_\sigma H_{\mu_2})$$

by

$$\pi(t)U = U\pi'(t) \quad \text{for all } t \in E_1 \otimes_c E_2.$$

We claim that π is an extension of $\mu_1 \otimes \mu_2$; to show this let $s \in E_1 \otimes^c E_2$ and $h \in H_0$, i.e. $h = \pi'(s_1)d$ for some $s_1 \in E_1 \otimes^c E_2$. Then

$$\mu_1 \otimes \mu_2(s)Uh = \mu_1 \otimes \mu_2(s)\mu_1 \otimes \mu_2(s_1)(a_1 \otimes a_2)$$

$$= \mu_1 \otimes \mu_2(ss_1)(a_1 \otimes a_2) = U[\pi'(ss_1)d]$$

$$= U\pi'(s)\pi'(s_1)d = U\pi'(s)h = \pi(s)Uh$$

for all $s \in E_1 \otimes^c E_2$ and $h \in H_0$. But $UH_0 = H$, which is dense in $H_{\mu_1} \otimes_\sigma H_{\mu_2}$. It follows that $\mu_1 \otimes \mu_2$ and π coincide wherever both are defined; consequently π extends $\mu_1 \otimes \mu_2$. This completes the proof.

Proposition 4.3 will be used to derive some corollaries from the following main theorem. Before stating and proving this theorem we need the following definitions and lemmas.

4.5. Suppose E_1, E_2 are Banach-*-algebras with approximate identities and suppose c is a cross norm on $E_1 \otimes E_2$ which is compatible with multiplication and involution (see 3.3 and 3.4). Suppose further that $C^*(E_i)$ denote the enveloping algebras and ε_i the enveloping mappings (see 2.8). Let

$$\varphi : E_1 \otimes E_2 \to C^*(E_1) \otimes_\nu C^*(E_2)$$

be the natural mapping, i.e.

$$\varphi\left(\sum x_i \otimes y_i\right) = \sum \varepsilon_1(x_i) \otimes \varepsilon_2(y_i)$$

for any $\sum x_i \otimes y_i \in E_1 \otimes E_2$. We require φ to be extendible to all of $E_1 \otimes_c E_2$. Since $C^*(E_1) \otimes_\nu C^*(E_i)$ is a C^*-algebra a necessary and sufficient condition for this to be possible is

(4.6) $\nu(\varphi(t)) \leqq c(t)$ for all $t \in E_1 \otimes^c E_2$,

[3, p. 7].

LEMMA 4.7. *Let E_1, E_2 be Banach-*-algebras with approximate identities and c a cross norm on $E_1 \otimes E_2$ compatible with multiplication and involution. Let π be a representation of $E_1 \otimes_c E_2$ in a Hilbert space H. With π we can associate representations π_i of E_i ($i = 1, 2$) in H with the properties*

(i) $\pi(u_a \otimes y) \to \pi_2(y)$ *strongly for any $y \in E_2$ and any approximate identity* $\{u_a\} \subset E_1$.

(ii) $\pi(x \otimes v_b) \to \pi_1(x)$ *strongly for any $x \in E_1$ and any approximate identity* $\{v_b\} \subset E_2$.

(iii) $\pi(x_1 \otimes x_2) = \pi_1(x_1)\pi_2(x_2) = \pi_2(x_2)\pi_1(x_1)$ *for all $x_i \in E_i$.*

The proof of Lemma 4.7 is identical with that of Proposition 1 of [10] and will therefore be omitted.

π_i is called the restriction of π to E_i. We can now prove the main result of this section.

THEOREM 4.8. *If E_1, E_2 are Banach-*-algebras with approximate identities and c a cross norm on $E_1 \otimes E_2$ which is compatible with multiplication and involution and which satisfies condition* (4.6), *then*

$$C^*(E_1 \otimes_c E_2) = C^*(E_1) \otimes_\nu C^*(E_2)$$

where = indicates the existence of a surjective isometric isomorphism.

Proof. We first show that $\nu(\varphi(s)) = |\varepsilon(s)|$ for any $s \in E_1 \otimes^c E_2$ where $|\cdot|$ is the norm in $C^*(E_1 \otimes_c E_2)$. $|\varepsilon(s)| \leq \nu(\varphi(s))$: Let π be a faithful representation of $C^*(E_1 \otimes_c E_2)$ and let ρ be the corresponding representation of $E_1 \otimes_c E_2$, i.e. $\rho = \pi \circ \varepsilon$, where ε is the enveloping mapping. Let ρ_i be the restriction of ρ to E_i, as defined in Lemma 4.7 and let π_i be the representation of $C^*(E_i)$ corresponding to ρ_i, i.e. $\rho_i = \pi_i \circ \varepsilon_i$, where ε_i is the enveloping mapping of E_i, $i = 1, 2$. For

$$t = \sum x_j \otimes y_j \in C^*(E_1) \otimes C^*(E_2)$$

define

$$\tau(t) = \sum \pi_1(x_j)\pi_2(y_j).$$

Since π_1 and π_2 are commuting representations (Lemma 4.7(iii)) τ is a representation of $C^*(E_1) \otimes C^*(E_2)$. Moreover, $\tau \circ \varphi$ is the restriction of ρ to $E_1 \otimes E_2$: if $\sum x_i \otimes y_i \in E_1 \otimes E_2$, then

$$\rho\left(\sum x_i \otimes y_i\right) = \sum \rho(x_i \otimes y_i) = \sum \rho_1(x_i)\rho_2(y_i) = \sum (\pi_1 \circ \varepsilon_1(x_i))(\pi_2 \circ \varepsilon_2(y_i))$$
$$= \tau\left(\sum \varepsilon_1(x_i) \otimes \varepsilon_2(y_i)\right) = \tau \circ \varphi\left(\sum x_i \otimes y_i\right).$$

Since

$$|\tau(x_1 \otimes x_2)| = |\pi_1(x_1)\pi_2(x_2)| \leq |\pi_1(x_1)| \, |\pi_2(x_2)| \leq |x_1| \, |x_2|$$

for all $x_i \in E_i$, $|\tau(t)| \leq \nu(t)$ for any $t \in C^*(E_1) \otimes C^*(E_2)$. Consequently, if $s \in E_1 \otimes^c E_2$, then

$$|\varepsilon(s)| = |\pi(\varepsilon(s))| = |\rho(s)| = |\tau \circ \varphi(s)| \leq \nu(\varphi(s)).$$

$\nu(\varphi(s)) \leq |\varepsilon(s)|$: To get this inequality we use condition (4.6). Let π be a faithful representation of $C^*(E_1) \otimes_\nu C^*(E_2)$. Then $\rho = \pi \circ \varphi$ is a representation of $E_1 \otimes_c E_2$. Let μ be the representation of $C^*(E_1 \otimes_c E_2)$ corresponding to ρ; if $s \in E_1 \otimes_c E_2$, then

$$\nu(\varphi(s)) = |\pi(\varphi(s))| = |\rho(s)| = |\mu \circ \varepsilon(s)| \leq |\varepsilon(s)|.$$

We have shown that

$$|\varepsilon(s)| = \nu(\varphi(s)) \quad \text{for all } s \in E_1 \otimes^c E_2.$$

Since $\varepsilon(E_1 \otimes^c E_2)$ is dense in $C^*(E_1 \otimes_c E_2)$ and $\varphi(E_1 \otimes^c E_2)$ is dense in $C^*(E) \otimes_\nu C^*(F)$ it follows that

$$|\varepsilon(s)| = \nu(\varphi(s)) \quad \text{for all } s \in E_1 \otimes_c E_2.$$

Moreover, $\varepsilon \circ \varphi^{-1}$ is seen to define an isometric isomorphism of $C^*(E_1) \otimes_\nu C^*(E_2)$ onto $C^*(E_1 \otimes_c E_2)$. This completes the proof.

Condition (4.6) is somewhat artificial looking. However, if we restrict the algebra under consideration, then this condition is satisfied by a natural class of norms on $E_1 \otimes E_2$. We have the following

Corollary 4.9 [26]. *If E_1 or E_2 is of type* I *and c is an ordinary norm compatible with multiplication and involution, then*

$$C^*(E_1 \otimes_c E_2) = C^*(E_1) \otimes_\alpha C^*(E_2).$$

Proof. Takesaki [23] has shown that if E_1 or E_2 is of type I then $\alpha = \nu$ (see 3.5). From the definition of α and Proposition 4.3 it follows that $\varepsilon_1 \otimes \varepsilon_2 \equiv \varphi$ can be extended to all of $E_1 \otimes_c E_2$, i.e. c satisfies condition (4.6). The result then follows from the above theorem.

We also note that condition (4.6) is satisfied by γ: if $t = \sum x_i \otimes y_i \in E_1 \otimes^\gamma E_2$, then

$$\nu(\varphi(t)) = \nu\left(\sum \varepsilon_1(x_i) \otimes \varepsilon_2(y_i)\right)$$

$$\leq \sum \nu(\varepsilon_1(x_i) \otimes \varepsilon_2(y_i))$$

$$\leq \sum |\varepsilon_1(x_i)| \, |\varepsilon_2(y_i)| \leq \sum |x_i| \, |y_i|.$$

This being true for any representative $\sum x_i \otimes y_i$ of t we get

$$\nu(\varphi(t)) \leq \inf \sum |x_i| \, |y_i| = \gamma(t)$$

for all $t \in E_1 \otimes E_2$.

It follows that Theorem 3 [16] is also a corollary of Theorem 4.8.

5. Spectra of tensor products.

5.1. In this section we study the relationships between $\hat{E}_1$, $\hat{E}_2$ and $(E_1 \otimes_c E_2)^\wedge$. Proposition 4.3 suggests that $\hat{E}_1 \times \hat{E}_2 \subset (E_1 \otimes_c E_2)^\wedge$ in a natural fashion. We show that this is the case and investigate the problem of equality of $\hat{E}_1 \times \hat{E}_2$ and $(E_1 \otimes_c E_2)^\wedge$.

5.2. Our first result concerns the embedding of $\hat{E}_1 \times \hat{E}_2$ into $(E_1 \otimes_c E_2)^\wedge$.

Recall that if E is a Banach-*-algebra with approximate identity then $\hat{E}$ and $[C^*(E)]^\wedge$ are homeomorphic (Proposition 2.10). This fact will be of help to us in the following.

We shall use the following notation: if S_i, V_i $(i=1, 2)$ are sets and $f_i: S_i \to V_i$ are functions then $f_1 \times f_2: S_1 \times S_2 \to V_1 \times V_2$ is defined by

$$f_1 \times f_2(s_1, s_2) = (f_1(s_1), f_2(s_2)) \quad \text{for all } (s_1, s_2) \in S_1 \times S_2.$$

We also note the following fact the proof of which is straightforward.

Lemma 5.3. *Let S_i, V_i $(i=1, 2)$ be topological spaces and $f_i: S_i \to V_i$ be open mappings. Then $f_1 \times f_2: S_1 \times S_2 \to V_1 \times V_2$ is open.*

Proposition 5.4. *Let E_1, E_2 be Banach-*-algebras with approximate identities and c an ordinary norm compatible with multiplication and involution on $E_1 \otimes E_2$. For every $\bar\mu_1 \in \hat{E}_1$, $\bar\mu_2 \in \hat{E}_2$ define G by $G(\bar\mu_1, \bar\mu_2) = (\mu_1 \otimes \mu_2)^\wedge$. Then*
   (i) $G(\hat{E}_1 \times \hat{E}_2) \subset (E_1 \otimes_c E_2)^\wedge$.

(ii) *G is 1-1.*

(iii) *G is continuous.*

(iv) *if c also satisfies condition* (4.6), *then* G^{-1} *is continuous.*

Proof. We first note that G is well defined; in fact if $\mu_1' \cong \mu_1''$ and $\mu_2' \cong \mu_2''$, then $\mu_1' \otimes \mu_2' \cong \mu_1'' \otimes \mu_2''$: suppose

$$\mu_i'(x_i)U_i h_i = U_i \mu_i''(x_i) h_i$$

for all $h_i \in H_{\mu_i''}$ and all $x_i \in E_i$ $(i=1, 2)$ where $U_i: H_{\mu_i''} \to H_{\mu_i'}$ $(i=1,2)$ is an isometry onto. $U_1 \otimes U_2$ defines an isometry of $H_{\mu_1''} \otimes_\sigma H_{\mu_2''}$ onto $H_{\mu_1'} \otimes_\sigma H_{\mu_2'}$. A straightforward algebraic manipulation and a simple continuity argument then show that

$$\mu_1' \otimes \mu_2'(t)(U_1 \otimes U_2)(h) = U_1 \otimes U_2(\mu_1'' \otimes \mu_2'')(t)h$$

for every $t \in E_1 \otimes_c E_2$ and every $h \in H_{\mu_1''} \otimes_\sigma H_{\mu_2''}$.

If $\mu_i \in \bar{\mu}_i \in \hat{E}_i$ $(i=1, 2)$, then $w^*[\mu_i(E_i)] = L(H_{\mu_i})$ (Proposition 2.3) so that

$$w^*[\mu_1 \otimes \mu_2(E_1 \otimes_c E_2)] = w^*[L(H_{\mu_1}) \otimes L(H_{\mu_2})]$$
$$= L(H_{\mu_1} \otimes_\sigma H_{\mu_2}) \qquad [2, \text{p. } 28].$$

This shows that $(\mu_1 \otimes \mu_2)^\wedge \in (E_1 \otimes_c E_2)^\wedge$.

To prove (ii) let $\bar{\mu}_i', \bar{\mu}_i'' \in \hat{E}_i$ be chosen so that $(\mu_1' \otimes \mu_2')^\wedge = (\mu_1'' \otimes \mu_2'')^\wedge \in (E_1 \otimes_c E_2)^\wedge$. We first observe that if $\pi_i^{(j)}$ is the restriction of $\mu_1^{(j)} \otimes \mu_2^{(j)}$ to E_i $(i, j=1, 2)$, then $\pi_i^{(j)}$ is a multiple of $\mu_i^{(j)}$. This is true since $\pi_1^{(j)} = \mu_1^{(j)} \otimes I_1^{(j)}$ where $I_1^{(j)}$ is the identity on $H_{\mu_2^{(j)}}$ and since a similar relationship holds between $\pi_2^{(j)}$ and $\mu_2^{(j)}$. If $\mu_1' \otimes \mu_2' \cong \mu_1'' \otimes \mu_2''$ then $\pi_i' \cong \pi_i''$: suppose that for some isometry U we have

$$U\mu_1' \otimes \mu_2'(t)h = \mu_1'' \otimes \mu_2''(t)Uh$$

for all $t \in E_1 \otimes_c E_2$ and $h \in H_{\mu_1' \otimes \mu_2'}$. Then for all $x_1 \in E_1$ and $h \in H_{\mu_1' \otimes \mu_2'}$ we have

$$U\pi_1'(x_1)h = U \lim_\beta (\mu_1'(x_1) \otimes \mu_2'(v_\beta))h$$
$$= \lim_\beta (\mu_1'' \otimes \mu_2''(x_1 \otimes v_\beta)Uh)$$
$$= \lim_\beta (\mu_1''(x_1) \otimes \mu_2''(v_\beta)(Uh))$$
$$= \pi_1''(x_1)Uh$$

where $\{v_\beta\}$ is an approximate identity for E_2. This shows $\pi_1' \cong \pi_1''$. Similarly $\pi_2' \cong \pi_2''$. The equivalence of π_i' and π_i'' implies the quasi-equivalence of μ_i' and μ_i'' [3, 5.3]. But two topologically irreducible quasi-equivalent representations are equivalent [3, 5.3.3]. This proves (ii).

To prove (iii) let $\delta_i: P(E_i) \to \hat{E}_i$ be the open continuous canonical mappings (see Proposition 2.10) and let $f \equiv \delta_1 \times \delta_2$. Then f is open and continuous (Lemma 5.3). If we define $g: P(E_1) \times P(E_2) \to P(E_1 \otimes_c E_2)$ by $g(p_1, p_2) = p_1 \otimes p_2$ (this

mapping was considered in Lemma 4.4) then we have the following commutative diagram

$$P(E_1) \times P(E_2) \xrightarrow{\; g \;} P(E_1 \otimes_c E_2)$$
$$\left\downarrow f \qquad\qquad\qquad \delta \right\downarrow$$
$$\hat{E}_1 \times \hat{E}_2 \xrightarrow{\quad G \quad} (E_1 \otimes_c E_2)\widehat{\;}.$$

Since f is open and continuous G is continuous iff $G \circ f$ is continuous [13, Theorems 8, 9, p. 95]. Moreover, $G \circ f = \delta \circ g$ and δ open and continuous imply that it suffices to show that g is continuous. But this is a simple consequence of the facts that pure states are uniformly bounded and $E_1 \otimes^c E_2$ is dense in $E_1 \otimes_c E_2$.

In the proof of (iv) we shall make use of Theorem 4.8. With the additional assumption on c we know that $C^*(E_1 \otimes_c E_2) = C^*(E_1) \otimes_v C^*(E_2)$. Since

$$[C^*(E_1 \otimes_c E_2)]\widehat{\;} = (E_1 \otimes_c E_2)\widehat{\;} \quad \text{and} \quad \hat{E}_i = [C^*(E_i)]\widehat{\;} \qquad (i = 1, 2)$$

($=$ indicates existence of a homeomorphism) it is clear that it suffices to show that the mapping $\mu \otimes v \to \mu$ from (a subset of) $[C^*(E_1 \otimes_c E_2)]\widehat{\;}$ into $[C^*(E_1)]\widehat{\;}$ is continuous. And this can be done by means of a slight variation of an argument due to Wulfsohn [25, Lemma 16].

Every open set in the spectrum of a C^*-algebra A is of the form

$$\hat{A}^I = \{\bar{\pi} \in \hat{A}; \pi|_I \not\equiv 0\}$$

where I is a closed 2-sided ideal [3, p. 62]. Let $G_1 \colon [C^*(E_1)]\widehat{\;} \times [C^*(E_2)]\widehat{\;} \to [C^*(E_1 \otimes_c E_2)]\widehat{\;}$ denote the mapping corresponding to G (see Proposition 4.3) and $h_i \colon [C^*(E_1)]\widehat{\;} \times [C^*(E_2)]\widehat{\;} \to [C^*(E_i)]\widehat{\;}$ the projection. Let $[C^*(E_1)]\widehat{\;}^I$ be an open set in $[C^*(E_1)]\widehat{\;}$. Its pre-image in $[C^*(E_1 \otimes_c E_2)]\widehat{\;}$ is $G_1(h_1^{-1}[C^*(E_1)]\widehat{\;}^I)$.

Note that $(I \otimes C^*(E_2), v)$, the v-closure of the algebraic tensor product $I \otimes C^*(E_2)$ is a closed 2-sided ideal in $C^*(E_1 \otimes_c E_2)$. By the above, to show that G_1^{-1} is continuous it suffices to identify $G_1(h_1^{-1}[C^*(E_1)]\widehat{\;}^I)$ with

$$[C^*(E_1 \otimes_c E_2)]\widehat{\;}^{(I \otimes C^*(E_2), v)} \cap G_1([C^*(E_1)]\widehat{\;} \times [C^*(E_2)]\widehat{\;}).$$

If $(\mu \otimes v)\widehat{\;} \in G_1(h_1^{-1}[C^*(E_1)]\widehat{\;}^I)$, then $\mu \in \bar{\mu}$ does not vanish on I and consequently $\mu \otimes v$ does not vanish on $I \otimes C^*(E_2)$ so

$$(\mu \otimes v)\widehat{\;} \in [C^*(E_1 \otimes_c E_2)]\widehat{\;}^{(I \otimes C^*(E_2), v)} \cap G_1([C^*(E_1)]\widehat{\;} \times [C^*(E_2)]\widehat{\;}).$$

Conversely, if $(\mu \otimes v)\widehat{\;}$ belongs to this last mentioned set we must show $\bar{\mu} \in [C^*(E_1)]\widehat{\;}^I$. Since if $\mu \in \bar{\mu}$ and $v \in \bar{v}$, $\mu \otimes v|_{(I \otimes C^*(E_2), v)} = \mu|_I \otimes v$ where the representation on the right-hand side is taken in the sense of Proposition 4.3 and since $\mu \otimes v|_{(I \otimes C^*(E_2), v)} \neq 0$ it follows that $\mu|_I \not\equiv 0$. This completes the proof.

Because of this proposition it is natural to investigate when G is an homeomorphism, i.e. is onto. As it turns out, a sufficient condition is that E_1 or E_2 be of type I. This condition is also necessary if E_1 and E_2 are separable. These facts can

be proved directly. However, we shall make some shortcuts by appealing to existing results via the enveloping algebras.

THEOREM 5.5. *If E_1, E_2 are Banach-∗-algebras with approximate identities and one of them is of type* I *and if c is an ordinary norm on $E_1 \otimes E_2$ compatible with multiplication and involution, then*

$$\hat{E}_1 \times \hat{E}_2 = (E_1 \otimes_c E_2)\widehat{},$$

i.e. G as defined in Proposition 5.4 is a homeomorphism.

Proof. $C^*(E_1 \otimes_c E_2) = C^*(E_1) \otimes_\alpha C^*(E_2)$ (Corollary 4.9). The conclusion then follows from [25, Lemma 13] and Proposition 2.10 (since E_1, say, is of type I iff $C^*(E_1)$ is of type I).

As a partial converse we have the following

THEOREM 5.6. *Let E_1, E_2 be separable Banach-∗-algebras with approximate identities and c a cross norm on $E_1 \otimes E_2$ compatible with multiplication and involution and satisfying condition* (4.6). *If $G: \hat{E}_1 \times \hat{E}_2 \to (E_1 \otimes_c E_2)\widehat{}$ defined in Proposition 5.4 is onto, then E_1 or E_2 is of type* I.

Proof. Suppose E_i is separable and that $S_i = \{x_n^i\}_{n=1}^\infty$ is a dense subset; then $\{\varepsilon_i(x_n^i)\}$ is dense in $C^*(E_i)$: given $t_i \in C^*(E_i)$ and $\varepsilon > 0$ there is $s_i \in E_i$ so that $|t_i - \varepsilon_i(s_i)| < \varepsilon/2$ and there is $x_{n_i}^i \in S_i$ so that $|s_i - x_{n_i}^i| < \varepsilon/2$. Then

$$|t_i - \varepsilon_i(x_{n_i}^i)| \leqq |t_i - \varepsilon_i(s_i)| + |\varepsilon_i(s_i) - \varepsilon_i(x_{n_i}^i)|$$
$$\leqq |t_i - \varepsilon_i(s_i)| + |s_i - x_{n_i^i}| < \varepsilon.$$

Evidently, the fact that G is onto implies that $G_1: [C^*(E_1)]\widehat{} \times [C^*(E_2)]\widehat{} \to [C^*(E_1 \otimes_c E_2)]\widehat{} = [C^*(E_1) \otimes_\nu C^*(E_1)]\widehat{}$ is onto. From Theorem 2 [17] we conclude that $C^*(E_1)$ or $C^*(E_2)$ is of type I and hence that E_1 or E_2 is of type I. This completes the proof.

6. ∗-semisimplicity.

6.1. In this section we consider the problem of whether ∗-semisimplicity (defined below) of E_1, E_2 is passed on to $E_1 \otimes_c E_2$ for a suitable norm c. This problem appears quite difficult and we present only a partial solution in the form of a necessary and sufficient condition that a tensor product of ∗-semisimple Banach-∗-algebras be ∗-semisimple. This condition is interesting in so far as it indicates a similarity between ∗-semisimplicity and the more general concepts semisimplicity and strong semisimplicity (for definitions, see [18, p. 55 and 59]): Gelbaum [6] and Gil de Lamadrid [8] show that if A and B are commutative semisimple Banach algebras and c is an ordinary norm so that $A \otimes_c B$ is a Banach algebra then $A \otimes_c B$ is semisimple iff the natural mapping of $A \otimes_c B$ into $A \otimes_\lambda B$ (to be defined) is 1-1. The same result holds for $\gamma = c$ and strong semisimplicity replacing semisimplicity when only one of the algebras is assumed commutative [26]. Here we

show that the same condition connects ∗-semisimplicity of E_1 and E_2 and of $E_1 \otimes_c E_2$ for a wide class of Banach-∗-algebras.

6.2. Let A be a ∗-algebra. The intersection of the kernels of all ∗-representations of A on Hilbert space is called the ∗-radical R^*. If $R^* = \{0\}$ then A is ∗-semisimple. This definition of R^* is a slight extension of that given in 2.8.

6.3. Suppose E_1 and E_2 are normed vector spaces and c is an ordinary norm on $E_1 \otimes E_2$, i.e. $\lambda \leqq c \leqq \gamma$. Since $\lambda \leqq c$ the identity mapping $E_1 \otimes E_2 \to E_1 \otimes_\lambda E_2$ has a unique extension to $E_1 \otimes_c E_2$, i.e. we can define

$$\tau_c \colon E_1 \otimes_c E_2 \to E_1 \otimes_\lambda E_2.$$

τ_c is a homomorphism. If E_1, E_2 are normed algebras, then τ_c is an algebra homomorphism. If τ_c is an isomorphism we also say that c is a *faithful* norm.

6.4. The main result of this section is that faithfulness of c is necessary and sufficient for the tensor product of ∗-semisimple Banach-∗-algebras E_1, E_2 to be ∗-semisimple, at least when E_1 or E_2 belongs to a rather wide class of algebras. This class is the class of Banach-∗-algebras for which $C^*(E)$ satisfies Takesaki's condition (T) [23]. As mentioned in 3.5 $C^*(E)$ satisfies condition (T) iff $C^*(E) \otimes_\nu B = C^*(E) \otimes_\alpha B$ for any C^*-algebra B. We say E satisfies condition (T) if $C^*(E)$ does.

For the proof a few preliminary facts are needed. The proof of the first of these is immediate from the definition of the enveloping algebra and will be omitted.

LEMMA 6.5. *If E is a Banach-∗-algebra with enveloping algebra $C^*(E)$, then E is ∗-semisimple iff the enveloping mapping $\varepsilon \colon E \to C^*(E)$ is 1-1.*

LEMMA 6.6. *α is faithful; i.e. if E and F are C^*-algebras, then $\tau_\alpha \colon E \otimes_\alpha F \to E \otimes_\lambda F$ is 1-1.*

Proof. Suppose $\tau_\alpha(t) = 0$; from the definition of the λ-norm it follows that $x' \otimes y'(t) = 0$ for any $x' \in E_*$ and $y' \in F_*$. By the definition of α we can identify $E \otimes_\alpha F$ with a sub-∗-algebra of $L(H_1 \otimes_\sigma H_2)$ for suitable Hilbert spaces H_1 and H_2. To show that $t = 0$ it suffices to show that $(th_1 \mid h_2) = 0$ for any $h_1, h_2 \in H_1 \otimes_\sigma H_2$ where $(\cdot \mid \cdot)$ denotes the inner product; in fact, it suffices to consider $h_1, h_2 \in H_1 \otimes H_2$ and hence, since $(\cdot \mid \cdot)$ is linear in both entries, to show that

$$(t(x_1 \otimes y_1) \mid (x_2 \otimes y_2)) \quad \text{for any } x_1, x_2 \in H_1, y_1, y_2 \in H_2.$$

Pick $x_1, x_2 \in H_1$ and $y_1, y_2 \in H_2$. $f(\cdot) = (\cdot x_1 | x_2)$ defines a continuous linear functional on $L(H_1)$ and hence on E; likewise, $g(\cdot) = (\cdot y_1 | y_2)$ is a continuous linear functional on F. An easy computation will show that

$$(t(x_1 \otimes y_1) \mid (x_2 \otimes y_2)) = f \otimes g(t) = 0.$$

This proves the lemma.

REMARK 6.7. A similar result does not hold for ν. For if $\tau_\nu \colon E \otimes_\nu F \to E \otimes_\lambda F$ is 1-1 then $\tau_{\nu\alpha} \colon E \otimes_\nu F \to E \otimes_\alpha F$ is 1-1 (since $\tau_\nu = \tau_\alpha \circ \tau_{\nu\alpha}$); and since $E \otimes_\nu F$ and

$E \otimes_\alpha F$ are C^*-algebras this implies that $\nu = \alpha$. This, however, is not generally true (see [23]). It is this fact that forces us to limit ourselves to Banach-$*$-algebras satisfying condition (T).

6.8. Since $\varepsilon: E \to C^*(E)$ is a bounded linear mapping it defines a dual mapping $\varepsilon_*: C^*(E)_* \to E_*$. In fact, if $a' \in C^*(E)_*$, then $\varepsilon_*(a') = f_{a'} \in E_*$ is defined by $f_{a'}(x) = a'(\varepsilon(x))$ for all $x \in E$.

We are now ready to state and prove

THEOREM 6.9. *Suppose E_1 and E_2 are Banach-$*$-algebras with approximate identities and that one of them satisfies condition (T). If c is an ordinary norm compatible with multiplication and involution on $E_1 \otimes E_2$ then $E_1 \otimes_c E_2$ is $*$-semisimple iff c is faithful.*

Proof. Suppose first that $E_1 \otimes_c E_2$ is $*$-semisimple. Choose $t_0 \in E_1 \otimes_c E_2$ such that $\tau_c(t_0) = 0$ where $\tau_c: E_1 \otimes_c E_2 \to E_1 \otimes_\lambda E_2$ is the canonical mapping (6.3). Let τ_α denote the canonical mapping of $C^*(E_1) \otimes_\alpha C^*(E_2)$ into $C^*(E_1) \otimes_\lambda C^*(E_2)$. Then by Lemma 6.6 α is faithful.

Now, $\tau_c(t_0) = 0 \Rightarrow x' \otimes y'(t_0) = 0$ for any $x' \in E_{1*}$, $y' \in E_{2*}$; in particular,

$$(6.10) \qquad\qquad [(\varepsilon_{E_1})_*(a') \otimes (\varepsilon_{E_2})_*(b')](t_0) = 0$$

for any $a' \in C^*(E_1)_*$, $b' \in C^*(E_2)_*$ (see 6.8). From (6.8) and (6.10) it follows that

$$((a' \circ \varepsilon_{E_1}) \otimes (b' \circ \varepsilon_{E_2}))(t_0) = 0$$

for any $a' \in C^*(E_1)_*$, $b' \in C^*(E_2)_*$ and hence that $a' \otimes b'(\varepsilon_{E_1} \otimes \varepsilon_{E_2})(t_0) = 0$ for any $a' \in C^*(E_1)_*$, $b' \in C^*(E_2)_*$. Since E_1 or E_2 satisfies condition (T), the proof of Corollary 4.9 enables us to conclude that $\lambda(\tau_\alpha(\varepsilon(t_0))) = 0$. By the above this implies that $\varepsilon(t_0) = 0$ and hence (Lemma 6.5) that $t_0 = 0$ i.e. τ_c is 1-1.

Conversely, suppose c is faithful and that $s_0 \in E_1 \otimes_c E_2$ is chosen so that $s_0 \neq 0$; then $\tau_c(s_0) = t_0 \neq 0$. As is known [19, I–III] any element t in $E_1 \otimes_\lambda E_2$ defines a continuous linear transformation $T_t: E_{1*} \to E_2$ and also $S_t: E_{2*} \to E_1$.

If t is a finite tensor in $E_1 \otimes_\lambda E_2$ and $\sum x_i \otimes y_i$ is a representative of t, then

$$T_t(x') = \sum x'(x_i) y_i \quad \text{for all } x' \in E_{1*}.$$

This defines T_t for every t in a dense subspace of $E_1 \otimes_\lambda E_2$. The definition is completed by use of the customary argument of extension by continuity. S_t is defined similarly.

Consider T_{t_0} and S_{t_0}. $t_0 \neq 0 \Rightarrow T_{t_0} \neq 0 \Rightarrow$ there is $x_0' \in E_{1*}$ such that $0 \neq T_{t_0}(x_0') \in E_2$. Since E_2 is $*$-semisimple, $\varepsilon_{E_2}(T_{t_0}(x_0')) \neq 0$ (Lemma 6.5); hence there is $b_0' \in C^*(E_2)_*$ such that

$$b_0'(\varepsilon_{E_2}(T_{t_0}(x_0'))) \neq 0.$$

Let $f_{b_0'} = (\varepsilon_{E_2})_*(b_0')$ (see 6.8). Now consider $S_{t_0}: E_{2*} \to E_1$. S_{t_0} is not zero; in fact, it is easy to see that

$$x_0'(S_{t_0}(f_{b_0'})) = b_0'(\varepsilon_{E_2}(T_{t_0}(x_0'))) \neq 0$$

where x_0' is the functional chosen above. From the *-semisimplicity of E_1 it follows that there is $a_0' \in C^*(E_1)_*$ such that

(6.11) $a_0'(\varepsilon_{E_1}(S_{t_0}(f_{b_0'}))) \neq 0.$

But a straightforward calculation will show that the left-hand side of (6.11) equals

$$(a_0' \otimes b_0')(\varepsilon_{E_1} \otimes \varepsilon_{E_2}(s_0)),$$

thus showing that $\varepsilon_{E_1} \otimes \varepsilon_{E_2}(s_0) \neq 0$ and therefore $\varepsilon(s_0) \neq 0$. We have shown that ε is 1-1 and consequently (6.5) that $E_1 \otimes_c E_2$ is *-semisimple.

The following converse result holds

PROPOSITION 6.10. *If E_1, E_2 are Banach-*-algebras with approximate identities and c an ordinary norm such that $E_1 \otimes_c E_2$ is a Banach-*-algebra, then E_1 and E_2 are *-semisimple if $E_1 \otimes_c E_2$ is *-semisimple.*

Proof. Let $0 \neq x_i \in E_i$; since c is a crossnorm $x_1 \otimes x_2 \in E_1 \otimes_c E_2$ is not zero. Therefore there is a representation π of $E_1 \otimes_c E_2$ such that $\pi(x_1 \otimes x_2) \neq 0$. Letting π_1 and π_2 denote the restrictions of π to E_1 and E_2 respectively, and using Lemma 4.7 we conclude that since $\pi_1(x_1)\pi_2(x_2) = \pi(x_1 \otimes x_2)$, $\pi_1(x_1) \neq 0$ and $\pi_2(x_2) \neq 0$. It follows that E_1 and E_2 are *-semisimple.

Grothendieck [14, p. 90] has shown that if G is a locally compact Hausdorff space and B is a Banach space, then the space of all continuous functions vanishing at infinity from G into B, $C(G, B)$ with the sup norm is isometrically isomorphic to $C(G) \otimes_\lambda B$ where $C(G)$ denotes the space of C-valued continuous functions vanishing at infinity. It is not difficult to see that this identification also preserves the natural multiplications if B—and hence $C(G, B)$—is a Banach algebra. Moreover, if B is a Banach-*-algebra, then the above isometry can easily be shown to preserve the natural involutions.

On the basis of these remarks we can state the following corollary, the proof of which is obvious from the above and Theorem 6.9. (It should be noted that commutative algebras satisfy condition (T) [23, Theorem 1].)

COROLLARY 6.11. *Let G be a locally compact T_2 space and B a Banach-*-algebra with approximate identity. Let $C(G, B)$ be the algebra of all continuous functions from G to B vanishing at infinity with the sup norm and involution $f^*(g) = (f(g))^*$ for any $f \in C(G, B)$. $C(G, B)$ is *-semisimple if and only if B is *-semisimple.*

Proof. The proof is clear from the above, since $C(G)$ has an approximate identity.

Corollary 6.10 can be proved directly. We have included it here primarily to introduce the identification $C(G, B) = C(G) \otimes_\lambda B$. This identification is also valid when $C(G)$ is not commutative (e.g. when we use convolution multiplication) and will be used again in §7.

7. **Examples.** In this section we apply the above theory to vector-valued group algebras over a compact T_2 group G. But first let us consider the numerical-valued case.

7.1. *Complex-valued group algebras.* Let G be a compact topological group in which the topology is Hausdorff and let m denote the Haar measure on G (since G is compact, m is left—as well as right—invariant, i.e. G is unimodular). We normalize m so that $m(G)=1$.

Kaplansky [12] shows that if $1 \leqq p < \infty$ then $L^p(G)$, the space of C-valued functions f on G for which

$$|f|_p = \left[\int |f(g)|^p \, dm(g)\right]^{1/p} < \infty,$$

is a semisimple dual Banach-*-algebra. Multiplication in $L^p(G)$ is defined by convolution, i.e. if $f_1, f_2 \in L^p(G)$, then $f_1 * f_2$ is defined by

$$(f_1 * f_2)(g) = \int f_1(gg_1)f_2(g_1^{-1}) \, dm(g_1).$$

Involution is defined by

(7.2) $f^*(g) = f(g^{-1})^*$ for $f \in G$

where the latter * denotes complex conjugation.

We shall show that

$$C^*(L^{p_1}(G)) = C^*(L^{p_2}(G)), \qquad p_1, p_2 \in [1, \infty).$$

This will be accomplished by showing that

(7.3) $[L^{p_1}(G)]^\wedge = [L^{p_2}G)]^\wedge, \qquad p_1, p_2 \in [1, \infty).$

(7.3) is to be interpreted as a symbolic expression of the existence of a particular 1-1 correspondence onto. We shall return to this point shortly. First, it is well known that the compactness of G implies that if $\infty > p \geqq q \geqq 1$, then

(7.4) $L^p(G) \subset L^q(G);$

in fact, $L^p(G)$ is an ideal in $L^q(G)$. Moreover, if $C^c(G)$ denotes the algebra of all continuous complex-valued functions on G with sup-norm, convolution multiplication and involution defined by (7.2) then

(7.5) $C^c(G) \subset L^p(G), \qquad \infty > p \geqq 1$

as an ideal. We denote the norm in $L^p(G)$ by $|\cdot|_p$ and in $C^c(G)$ by $|\cdot|_\infty$ and have, since the measure is normalized on G, that

(7.6) $|\cdot|_\infty \geqq |\cdot|_p \geqq |\cdot|_1$ for $p \in [1, \infty)$

[15, p. 430].

Because of the inclusions (7.4) and (7.5) and the norm relationship (7.6) it is clear that

$$[L^1(G)]^\wedge \subset [L^p(G)]^\wedge \subset [C^c(G)]^\wedge$$

for any p, $1 \leqq p < \infty$ in the sense that a topologically irreducible representation of $L^1(G)$ uniquely determines (by restriction) a topologically irreducible representation of $L^p(G)$ ($1 \leqq p < \infty$) which in turn uniquely determines (again by restriction) a topologically irreducible representation of $C^c(G)$. Consequently (7.3) can be proved by showing that

(7.7) $$[C^c(G)]^\wedge = [L^1(G)]^\wedge.$$

Again the equality means the existence of a special 1-1 correspondence onto; any topologically irreducible representation of $L^1(G)$ defines by restriction a topologically irreducible representation of $C^c(G)$; conversely, a topologically irreducible representation of $C^c(G)$ is (uniquely) extendible to a topologically irreducible representation of $L^1(G)$. We give an indication of the proof of this.

Since any unitary representation of G generates a representation of $L^1(G)$ [15, pp. 378–379], to show the validity of (7.7), it suffices to prove that if $\bar\pi \in [C^c(G)]^\wedge$ then $\pi \in \bar\pi$ generates a unitary representation of G. In [15, pp. 377–378] it is proved that any topologically irreducible representation of $L^1(G)$ induces a unitary representation of G. The same proof works to show that a topologically irreducible representation of $C^c(G)$ generates a unitary representation of G. For this reason, the proof will not be reproduced here. Because of these identifications we shall use $\hat{G}$ to denote the spectrum of $L^p(G)$, $1 \leqq p < \infty$ and the same symbol for $[C^c(G)]^\wedge$.

We can now complete the proof of the equation (7.3). According to the above discussion it suffices to prove that $C^*(L^1(G)) = C^*(C^c(G))$. Since $C^c(G)$ is $*$-semi-simple $C^*(C^c(G))$ can be interpreted as the completion of $C^c(G)$ with respect to the norm $|f|' = \sup_{\bar\pi \in [C^c(G)]^\wedge} |\pi(f)|$ (2.8). In the same vein $C^*(L^1(G))$ is the completion of $L^1(G)$ with respect to the norm $|f|'' = \sup_{\bar\pi \in [L^1(G)]^\wedge} |\pi(f)|$. It follows from (7.7) that $|f|' = |f|''$ for $f \in C^c(G)$. Since $C^c(G)$ is dense in $L^1(G)$ with respect to the 1-norm we can conclude that $C^*(C^c(G)) = C^*(L^1(G))$. We have proved

PROPOSITION 7.8. $C^*(L^{p_1}(G)) = C^*(L^{p_2}(G))$ for any $p_1, p_2 \in [1, \infty)$.

On the basis of Proposition 7.8 we shall denote the enveloping algebra of $L^1(G)$ by $C^*(G)$.

Finally, we observe that $L^p(G)$, $1 \leqq p < \infty$ is of type I, since $C^*(G)$ is of type I. A proof of the fact that $C^*(G)$ is of type I can be found in [3, 15.1.5, 4.2.4 and 5.5.2]. Since $L^p(G)$ is known to have approximate identities the theory developed in §§4, 5 and 6 applies to tensor products involving $L^p(G)$.

7.9. *Vector-valued group algebras.* Examples of such tensor products are provided by the spaces $B^p(G, A)$ where G is a compact Hausdorff group, A a Banach

algebra or Banach-∗-algebra and $1 \leqq p < \infty$. These spaces have been studied by Spicer [20], [21] and are defined as the spaces of functions $f: G \to A$ for which

$$\left[\int |f(g)|^p \, dm(g) \right]^{1/p} < \infty.$$

$B^p(G, A)$ is normed by defining

$$|f|_p = \left[\int |f(g)|^p \, dm(g) \right]^{1/p}.$$

Spicer [20, Chapter III, Theorem 1] shows that $B^p(G, A) = L^p(G) \otimes_p A$ where the norm on the tensor product is defined in the following fashion.

If $t = \sum x_i \otimes y_i$ is a finite tensor in $L^p(G) \otimes A$ consider the associated expression $\sum x_i(\cdot) y_i$ which defines a function from G into A. We define

$$p(t) = \left[\int \left| \sum x_i(g) y_i \right|^p \, dm(g) \right]^{1/p}.$$

This norm is shown [20, Chapter III, Proposition 3] to be an ordinary norm and compatible with multiplication. It is also compatible with involution.

With t as before

$$
\begin{aligned}
p(t^*) &= \left[\int \left| \sum x_i^*(g) y_i^* \right|^p \, dm(g) \right]^{1/p} \\
&= \left[\int \left| \sum \overline{x_i(g^{-1})} y_i^* \right|^p \, dm(g) \right]^{1/p} \\
&= \left[\int \left| \left(\sum x_i(g^{-1}) y_i \right)^* \right|^p \, dm(g) \right]^{1/p} \\
&= \left[\int \left| \sum x_i(g^{-1}) y_i \right|^p \, dm(g) \right]^{1/p} \\
&= \left[\int \left| \sum x_i(g) y_i \right|^p \, dm(g) \right]^{1/p} = p(t).
\end{aligned}
$$

Since the involution is isometric with respect to p on $L^p(G) \otimes^p A$, the same is the case on $L^p(G) \otimes_p A$.

If A is a Banach-∗-algebra with approximate identity then §§4 and 5 immediately give us the following structure theorems:

PROPOSITION 7.10. $C^*(B^p(G, A)) = C^*(G) \otimes_\alpha C^*(A)$ and

PROPOSITION 7.11. $[B^p(G, A)]^\frown = \hat{G} \times \hat{A}$.

Finally, we prove

PROPOSITION 7.12. *If A is a Banach-∗-algebra with approximate identity and G is a compact T_2 group, then $B^p(G, A)$ is ∗-semisimple iff A is ∗-semisimple $[1 \leqq p < \infty]$.*

Proof. The proof makes use of an extension of Grothendieck's result concerning spaces of vector valued continuous functions. Spicer [20, pp. 38–39] observes that this result also implies that $C^c(G, A) = C^c(G) \otimes_\lambda A$.

First we note that Theorem 6.9 and Proposition 6.10 do not apply directly to the present situation since $C^c(G)$ does not have an approximate identity. However, an examination of the relevant parts of the proof of Theorem 6.9 shows that if we assume the cross norm c to be faithful then it follows that $\varepsilon_{E_1} \otimes \varepsilon_{E_2}$ is 1-1 and therefore that ε is 1-1 i.e. that $E_1 \otimes_c E_2$ is *-semisimple provided that E_1 and E_2 are (we are using the notation of Theorem 6.9). It follows from these observations that $C^c(G, A)$ is *-semisimple if A is *-semisimple.

Suppose A is *-semisimple. Then $C^c(G, A)$ is *-semisimple. We want to show that $B^p(G, A)$ is *-semisimple. Let $t \in B^p(G, A)$ and suppose $\pi(t) = 0$ for all

$$\bar\pi \in [B^p(G, A)]^\wedge = [C^c(G, A)]^\wedge.$$

By Proposition 3 of Chapter II in [20] we can choose a net $\{w_\nu\} \subset C^c(G, A)$ such that $|w_\nu|_1 \leq 1$ and $\{w_\nu\}$ otherwise acts as an approximate identity in $B^p(G, A)$. Since $C^c(G, A)$ is an ideal in $B^p(G, A)$ [20, p. 15] it follows that $w_\nu * t \in C^c(G, A)$ for all ν. Moreover $\pi(t) = 0 \Rightarrow \pi(w_\nu * t) = 0$ for any $\bar\pi \in [C^c(G, A)]^\wedge$ and for all ν. Since $w_\nu * t \to t$ and since the *-semisimplicity of $C^c(G, A)$ implies that $w_\nu * t = 0$ for all ν, it follows that $t = 0$, i.e. $B^p(G, A)$ is *-semisimple.

The other direction follows directly from the fact that A can be identified with a closed ideal in $C^c(G, A)$ (the constant functions).

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